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Symmetries of type N empty space-times possessing twisting geodesic rays

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Abstract. It is shown that if the geodesic ray congruence associated with an empty space–time of type N is *not* hypersurface orthogonal (that is the geodesic rays are twisting) then the space–time has at most one symmetry.

1. Introduction

The field equations for algebraically special empty space–times have been solved recently for various cases. One case for which explicit solutions have not yet been found is that of type N space–times with twisting geodesic rays. If any of these space–times were to exhibit several degrees of symmetry, then it should be possible to adapt the coordinate system to the symmetries and so simplify the field equations. For this reason the symmetries of these space–times are investigated here and the following theorem is proved.

Theorem. Empty space–times of type N with twisting geodesic rays have, at most, one symmetry.

If the symmetry is assumed to exist, then the resulting simplified field equations are, unfortunately, still too difficult to solve.

2. The field equations

The field equations for algebraically special empty space–times with twisting geodesic rays have been reduced to a canonical form by Kerr (1963). However, Kerr does not give in his paper the conditions for particular Petrov types, nor does he give the coordinate transformations which leave invariant the field equations. For this reason it is necessary to rederive the field equations for the type N empty space–times. This is best done using the null tetrad notation of Newman and Penrose (1962), together with the form of the field equations given by Newman *et al.* (1963). A short calculation shows that, using coordinates $(x^1, x^2, x^0, \bar{x}^0) \equiv (u, r, z, \bar{z})$, the metric can be written in terms of the null tetrad

$$l^i = \delta_2{}^i$$
$$n^i = \tfrac{1}{2}(D\dot{\bar{\Omega}} + \bar{D}\dot{\Omega})\delta_2{}^i - \delta_1{}^i$$

and

$$m^i = (r - \mathrm{i}\Delta)^{-1}\{\mathrm{i}\bar{D}\Delta\delta_2{}^i - \dot{\bar{\Omega}}(r + \mathrm{i}\Delta)\delta_2{}^i + \bar{\Omega}\delta_1{}^i - \delta_{\bar{0}}{}^i\}$$

where a dot denotes differentiation with respect to u, the operator D is defined by

$$D = \frac{\partial}{\partial z} - \Omega\frac{\partial}{\partial u} \tag{2.1}$$

and Δ is defined in terms of the complex function $\Omega(z, \bar{z}, u)$ by

$$2\mathrm{i}\Delta = \bar{D}\Omega - D\bar{\Omega}. \tag{2.2}$$

The function Ω satisfies the two field equations†

$$\bar{D}\frac{\partial}{\partial u}(D\Omega) = 0 \tag{2.3}$$

† The left-hand side of equation (2) in the paper by Kerr (1963) should be differentiated with respect to u.

and

$$\bar{D}\bar{D}D\Omega = DD\bar{D}\bar{\Omega} \ . \tag{2.4}$$

The non-zero spin coefficients (see Newman and Penrose (1962) for definitions) are

$$\rho = -(r+\mathrm{i}\Delta)^{-1}, \qquad \alpha = \dot{\Omega}(r+\mathrm{i}\Delta)^{-1}, \qquad \mu = \bar{D}\dot{\Omega}(r-\mathrm{i}\Delta)^{-1}$$

and

$$\nu = (r+\mathrm{i}\Delta)^{-1}\{\ddot{\Omega}(r-\mathrm{i}\Delta)+D\bar{D}\dot{\Omega}-2\dot{\Omega}\bar{D}\dot{\Omega}\}.$$

The only non-zero tetrad component of the curvature tensor is

$$\psi_4 = -(r+\mathrm{i}\Delta)^{-1}\frac{\partial^2(D\Omega)}{\partial u^2}.$$

The form of the metric is invariant under the coordinate transformation

$$u' = u+f(z, \bar{z}) \tag{2.5}$$

and also under the combined coordinate and tetrad transformation

$$z' = \int g(z)\,\mathrm{d}z, \qquad r' = r(g\bar{g})^{-1/2}, \qquad l^{i'} = (g\bar{g})^{1/2}l^i$$

$$n^{i'} = (g\bar{g})^{-1/2}n^i, \qquad m^i = \left(\frac{g}{\bar{g}}\right)^{1/2} m^i. \tag{2.6}$$

The function Ω transforms under (2.5) and (2.6) as

$$\Omega' = \Omega - \frac{\partial f}{\partial z} \tag{2.7}$$

and

$$\Omega' = \Omega g^{-1}. \tag{2.8}$$

3. Symmetries of the space–times

The symmetries of a space–time are generated by the solutions v_i of Killing's equation

$$v_{i;j}+v_{j;i} = 0.$$

The null tetrad components of this equation and of the first set of integrability conditions have been written out explicitly in a previous paper (Collinson and French 1967). The only equations required here are

$$v_{1;i}l^i = 0 \tag{3.1}$$

$$v_{2;i}l^i+v_{1;i}n^i = 0 \tag{3.2}$$

$$v_{3;i}l^i+\rho v_3 = 0 \tag{3.3}$$

$$v_{1;i}m^i+v_{3;i}l^i-\bar{\alpha}v_1+\bar{\rho}v_3 = 0 \tag{3.4}$$

and

$$\bar{v}_{3;i}m^i+v_{3;i}\bar{m}^i-(\mu+\bar{\mu})v_1+(\rho+\bar{\rho})v_2-\bar{\alpha}\bar{v}_3-\alpha v_3 = 0. \tag{3.5}$$

Equations (3.1), (3.2) and (3.3) can be solved to give

$$v_1 = v_1{}^0$$

$$v_2 = \dot{v}_1{}^0 r+v_2{}^0$$

and

$$v_3 = v_3{}^0(r+\mathrm{i}\Delta)$$

where $v_1{}^0$, $v_2{}^0$ and $v_3{}^0$ are functions independent of r.

Substituting these expressions into equations (3.4) and (3.5) yields

$$-\bar{D}v_1{}^0 = \dot{\bar{\Omega}}\, v_1{}^0 + 2\,\mathrm{i}\Delta v_3{}^0 \tag{3.6}$$

$$-\bar{D}\bar{v}_3{}^0 - Dv_3{}^0 - 2\dot{\bar{\Omega}}\bar{v}_3{}^0 - 2\dot{\Omega}v_3{}^0 = 2\dot{v}_1{}^0 \tag{3.7}$$

and

$$2\mathrm{i}\bar{v}_3{}^0\bar{D}\Delta - 2\mathrm{i}v_3{}^0D\Delta - 2\mathrm{i}\bar{v}_3{}^0\Delta\dot{\bar{\Omega}} + 2\mathrm{i}v_3{}^0\Delta\dot{\Omega} = (\bar{D}\dot{\Omega} + D\dot{\bar{\Omega}})v_1{}^0 + 2v_2{}^0 \tag{3.8}$$

where D is the operator defined by equation (2.1).

Let us assume first that $v_3{}^0 = 0$. Then, from equation (3.7), $v_1{}^0$ is independent of u. Differentiating (3.6) with respect to u yields $\ddot{\bar{\Omega}}v_1{}^0 = 0$. If $v_1{}^0$ is zero, then, from equation (3.8), $v_2{}^0$ is also zero and so Killing's equation does not admit a solution. If $\ddot{\bar{\Omega}}$ is zero, then ψ_4 is zero and the space-time is flat. Now, let us assume that $v_3{}^0 \neq 0$. Under the transformation (2.6), $v_3{}^{0'} = gv_3{}^0$, and so the transformation can be used to set $v_3{}^0 = \mathrm{i}$. Equation (3.7) then yields

$$v_1{}^0 = \mathrm{i}(\bar{\Omega} - \Omega) + v_1{}^0(z, \bar{z}).$$

Under the transformation (2.5)

$$v_1{}^0(z, \bar{z})' = v_1{}^0(z, \bar{z}) + \mathrm{i}\left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial f}{\partial z}\right)$$

and so the transformation can be used to set $v_1{}^0\,(z, \bar{z}) = 0$. Hence

$$v_1{}^0 = \mathrm{i}(\bar{\Omega} - \Omega) \qquad \text{and} \qquad v_3{}^0 = \mathrm{i}. \tag{3.9}$$

Substituting these into equation (3.8) gives $v_2{}^0$, and so Killing's equation admits a unique solution. This proves the theorem stated in § 1.

The transformations (2.5) and (2.6) have been used to adapt the coordinate system to the symmetry of the space-time. Substituting (3.9) into equation (3.6) yields a canonical form for Ω, namely

$$\Omega = \Omega(z + \bar{z}, u).$$

Hence when the space-time exhibits a symmetry the function Ω is reduced to a function of two variables, but, unfortunately, this is not a sufficient simplification to aid in the solution of the field equations (2.3) and (2.4).

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